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COMMENT

On integrals of combinations of solutions of second-order differential equations

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Abstract. It is shown how integrals of certain combinations of solutions to a second-order differential equation may be found in an elementary way. In particular, an integral involving Airy functions which has occurred in recent studies of the asymptotic behaviour of the gap in the Mathieu equation is evaluated.

The integral

$$J=\int_0^\infty \frac{\mathrm{Ai}(x)\mathrm{Bi}(x)}{(\mathrm{Ai}^2(x)+\mathrm{Bi}^2(x))^2}\,\mathrm{d}x \tag{1}$$

which occurred in Harrell's study (1981) of the asymptotics of the gaps in Mathieu's equation, was recently evaluated this using special formulae for the Airy functions (Wille and Vennik 1985). This comment presents a simpler procedure for evaluating the given integral which, in addition, applies to a wider class of integrals.

Let y_1 and y_2 be two linearly independent solutions to

$$y''=Q(x)y \tag{2}$$

and observe that their Wronskian $W\{y_1, y_2\}=c$, a non-zero constant.

Now consider the integral

$$\int f(y_1(x), y_2(x))\,\mathrm{d}x \tag{3}$$

where f is homogeneous of degree -2, i.e. $f(\alpha y_1, \alpha y_2)=\alpha^{-2}f(y_1, y_2)$. Then we have $f(y_1, y_2)=y_1^{-2}f(1, y_2/y_1)$ and thus

$$\begin{aligned}\int f(y_1(x), y_2(x))\,\mathrm{d}x &= \frac{1}{c}\int f(1, y_2/y_1)\frac{W\{y_1, y_2\}}{y_1^2}\,\mathrm{d}x\\ &= c^{-1}\int f(1, u)\,\mathrm{d}u \end{aligned}\tag{4}$$

where $u=y_2/y_1$ (and therefore $u'=W\{y_1, y_2\}/y_1^2$). Hence we can find an antiderivative for $f(y_1(x), y_2(x))$ just by integrating $f(1, u)$.

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As an example of the use of this approach consider the integral J given by equation (1). The method above applies directly, yielding (we take $u = \mathrm{Bi}/\mathrm{Ai}$ and use $W\{\mathrm{Ai}, \mathrm{Bi}\} = \pi^{-1}$ and $\mathrm{Bi}(0)/\mathrm{Ai}(0) = \sqrt{3}$ from Abramowitz and Stegun (1964), equations 10.4.10 and 10.4.4)

$$J = -\tfrac{1}{2}\pi[1 + (\mathrm{Bi}/\mathrm{Ai})^2]^{-1}|_0^\infty = \tfrac{1}{2}\pi[1 + (\mathrm{Bi}(0)/\mathrm{Ai}(0))^2]^{-1} = \tfrac{1}{8}\pi \tag{5}$$

in agreement with Avron and Simon (1981), Harrell (1981) and Wille and Vennik (1985). (Note, however, that Wille and Vennik's equation (10) lacks a minus sign.)

Alternatively, one can evaluate the integral which Harrell first encountered:

$$I = \int_0^\infty \frac{\mathrm{d}x}{(\mathrm{Ai}(x) - \mathrm{i}\mathrm{Bi}(x))^2}. \tag{6}$$

With $y_1(x) = \mathrm{Ai}(x) - \mathrm{i}\mathrm{Bi}(x)$ and $y_2(x) = \mathrm{Bi}(x)$ we have

$$\begin{aligned} I &= \pi \int_0^\infty \frac{y_1 y_2' - y_1' y_2}{y_1^2}\, \mathrm{d}x = \pi \frac{y_2}{y_1}\bigg|_0^\infty = \frac{\pi \mathrm{Bi}(x)}{\mathrm{Ai}(x) - \mathrm{i}\mathrm{Bi}(x)}\bigg|_0^\infty \\ &= \pi(\mathrm{i} - \sqrt{3})/4 \end{aligned} \tag{7}$$

so that $\mathrm{Im}(I) = \pi/4 = 2J$ as required. Note that the evaluation effected above can also be viewed as the method of reduction of order in a slightly disguised form. From that perspective one has

$$(\mathrm{Ai}(x) - \mathrm{i}\mathrm{Bi}(x)) \int_x^\infty \frac{\mathrm{d}x}{(\mathrm{Ai}(x) - \mathrm{i}\mathrm{Bi}(x))^2} = \mathrm{i}\pi\mathrm{Ai}(x) \tag{8}$$

which is equivalent to a variant of equation (7) above. Equation (8) provides the most efficient means of evaluating the integrals I and J and indeed is the method presented by Harrell in his paper.

As a further example of the use of this procedure the integral

$$\int \frac{\sin^2 x}{\cos^4 x + \cos x \sin^3 x}\, \mathrm{d}x \tag{9}$$

will now be evaluated. With $y_1 = \cos x$ and $y_2 = \sin x$ one obtains

$$\int \frac{y_2^2(y_1 y_2' - y_1' y_2)}{y_1^4 + y_1 y_2^3}\, \mathrm{d}x = \tfrac{1}{3} \ln|1 + (y_2/y_1)^3| + C \tag{10}$$

i.e. $\tfrac{1}{3}\ln|1 + \tan^3 x|$ is an antiderivative of the integrand above. Note that this provides a much simpler method of evaluating the given integral than the standard (and laborious) 'conversion to rational functions' approach via the substitution $z = \tan x/2$. On the other hand, the method presented here is not as widely applicable and even when applicable does not always work out as easily as the example above.

One can, of course, apply the method given here to integrals of the form (3) involving many of the other special functions of mathematical physics. For functions whose defining differential equation has a term in y' one could begin by making a change of variable to bring the equation to the form (2) and then proceed with the method as given or, alternatively, one could consider integrals like (3) but modified to contain a factor proportional to the Wronskian (which is easily found from Abel's identity).

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